Chapter 6

Test $\hat{K}\sigma$. The Basic Algorithm

In this chapter, the *double sigma test* explained in section 4.4 is adapted to fulfill the requirements of our counting scale. Then, the basic algorithm is presented and the optimal values for its parameters are calculated depending on the variance of the process. A solution is selected to satisfy the specifications for any value the variance of the process can take. Finally, the test is proved with some data series and results are shown.

6.1 Description of the algorithm

As it is said in section 4.4, the structure of the *double sigma test* could be suitable for the scale we are dealing with. However, it is necessary to adapt it to satisfy the problem specifications. More concretely, the test limits have to be changed from 2σ to a new value K to ensure the required significance level $\alpha = 0.01$.

The basics of the test performance are identical to those of the *double sigma test* and can be summarized in the next procedures list:

1. Estimation of the mean of the process $\hat{\mu}$ (i.e.: the initial average weight on the scale). Therefor, a list of measurements have to be taken for a long time period during which the weight on the scale should not change. Then the counter is set to zero.

2. Calculation of the *lower control limit* (*LCL*) and the *upper control limit* (*UCL*) depending on the chosen sample size N.

3. Reading of N weight values that make up the first sample. The sample is loaded into a queue of size N.

4. Reading a new data x_i and writing it into the queue.

5. Calculation of the new sample mean $\bar{x}_i$.

6. The new value is compared with the control limits:

   - If $\bar{x}_i > UCL$, then the counter is increased in a unit and the estimated process mean is increased in the weight of a unit: $\hat{\mu}^* = \hat{\mu} + 147gr$.

- If $\bar{x}_i < LCL$, then the counter is decreased in a unit and the estimated process mean is decreased in the weight of a unit: $\hat{\mu}^* = \hat{\mu} - 147gr$.
- Otherwise, $\hat{\mu}$ is corrected by filtering $\bar{x}_i$.

7. Values *UCL* and *LCL* are adapted to the new value of $\hat{\mu}$ keeping the same width of the control area.

8. Return to step 4.

A flow diagram implementing this algorithm is presented in figure 6.1.

6.2 Correction of $\hat{\mu}$

On figure 6.1, the block named "*Correction of* $\hat{\mu}$" corresponds to the third case at step 6 in the previous list. Its goal is to adapt the estimated value of the process mean ($\hat{\mu}$) by filtering $\bar{x}_i$ values. This is conceptually important due to the special behavior counting scales must have. As is has been said in the introduction, *counting scales must register and count the leaps on the reading* but not just the total weight. So, as we constantly adapt $\hat{\mu}$ by filtering $\bar{x}_i$, we enable this variable to take any value leaving predefined levels corresponding to undivided unitary-weight-increments. As a consequence, slow or small increments of the weight will not accumulate and they will not affect the units counting.

The filtering is implemented through a single real-pole:

$$\hat{\mu}_i = K\frac{z}{z-A}\bar{x}_i; \qquad A \leq 1.$$

For the filter to have unitary gain, we must set $K = 1 - A$. So we have:

$$\hat{\mu}_i = (1-A)\frac{z}{z-A}\bar{x}_i \tag{6.1}$$

That can be implemented through the incremental equation:

$$\hat{\mu}_i = A\hat{\mu}_{i-1} + (1-A)\bar{x}_i \tag{6.2}$$

The value of parameter A will determine the speed of the response. If the response is too fast, weight increments might never be detected, since reference $\hat{\mu}$ would undergo the same variations than the sample mean. In order to select an appropriate value for A, the response to the step has been studied. From (6.2), and by considering: $\bar{x}_i = \hat{\mu}_0 + \Delta\mu, \quad \forall i$; the response to the step can be calculated as:

$$\begin{aligned}
(\bar{x}_i - \hat{\mu}_i) &= (\bar{x}_i - \hat{\mu}_{i-1})A \\
(\bar{x}_i - \hat{\mu}_i) &= (\bar{x}_i - \hat{\mu}_0)A^i = \Delta\mu A^i \\
\frac{(\bar{x}_i - \hat{\mu}_i)}{\Delta\mu} &= A^i. \qquad (6.3)
\end{aligned}$$

As a criterion, it has been considered that, in the average delay time between the moment when an unit falls in the scale and the moment when it is detected (this delay time is a parameter calculated in this work), the assumed process mean should evolve less than 1% of the weight increment. This means:

$$\begin{aligned} A^{T_{delay}} &\geq 0.99 \\ \log_{10} A &\geq \frac{\log_{10} 0.99}{T_{delay}} \end{aligned} \tag{6.4}$$

Where T_{delay} is the already mentioned delay time expressed in number of samples.

Now, we present a result from chapter 7 in advance, just to show the value this parameter finally takes: $\bar{T}_{delay} \approx 15$samples. That leads to: $A \geq 0.9993$. The mostly used value in this work is:

$$\boxed{A = 0.9995}$$

6.3 Test parameters and their influence in the test response

The condition concerning the significance level of the test can be expressed as:

$$P(LCL \leq y \leq UCL) \geq \alpha \tag{6.5}$$

Where LCL and UCL are the *lower* and the *upper control limit* respectively, and $y = \bar{x}_n = \frac{1}{n}\sum_{i=1..n} x_i$.

In order to perform a general study that can easily be adapted to different conditions, the normalized distribution is used and, therefore, normalized values of UCL and LCL are now calculated. As it was said in section 5.4, the measurements distribution is considered to be symmetric, so selected test limits are also symmetric, resulting: $UCL - \mu = \mu - LCL = K$. Test limits can thus be expressed through a single normalized value as:

$$\begin{aligned} \hat{K} &= \tfrac{UCL-\mu}{\sigma} \\ \hat{K} &= \tfrac{\mu-LCL}{\sigma}. \end{aligned} \tag{6.6}$$

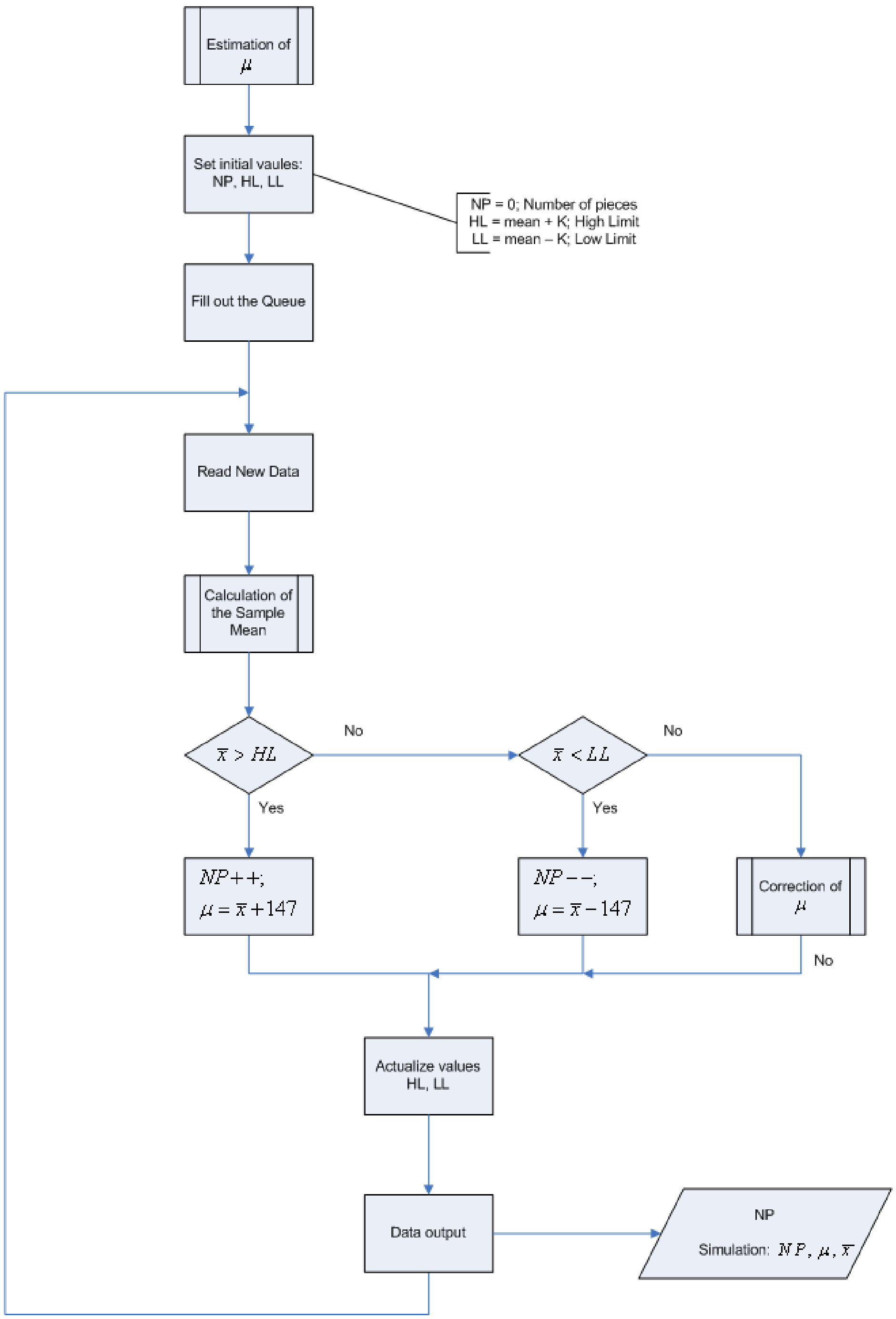
It is interesting to compare the new expression of the test limit, $K = \hat{K}\sigma$, with the value of the limits in the double sigma test, where $K = 2\sigma$. So,the double sigma test can be considered a particular case of this test where $\hat{K} = 2$.

To find the appropriate value of $\hat{K}$ for our test, expression (6.5) is written now as:

$$P(-\hat{K} \leq \hat{y} \leq \hat{K}) \geq \alpha \tag{6.7}$$

Where: $\hat{y} = \frac{y-\mu}{\sigma_y}$.

Figure 6.1: Flow diagram describing the algorithm for the test



Letting $D(0,1)$ denote the normalized empirical probability density function (PDF) calculated in chapter 5, we have:

$$P(-\hat{K} \leq D(0,1) \leq \hat{K}) \geq \alpha \tag{6.8}$$

Now, by numerically solving equation (6.8) with $\alpha = 0.01$, we obtain:

$$\boxed{\hat{K} = 3.3418}$$

Once we set the value of $\hat{K}$, the probability of *type I errors* is fixed to 1%, but we still have to choose a value for the sample size N. This additional degree of freedom is used to optimize the probability of *type II errors* (β). However, since there is no specification for this parameter, it is necessary to find a practical rule based on the common sense that tells us if the reached value of β is good enough for the test. That is the reason that the duration of *type II errors* (i.e., the number of consecutive *type II errors*) is used in this thesis as criterion instead of using directly the value of β.

The duration of type II errors (or the *delay time* of the test's response) is strongly related to β. The average value of the delay time[1] corresponds to the mean value of a PDF describing the first success of a sequence of *Bernoulli experiments* with *different* success probability each time (Similar to a *Pascale distribution*).

Dependency of the delay time on N

Low values of N lead to high values of σ and, as a consequence, to wide control areas. When the control band is too wide compared to the size of the step, it is easy that new measurements are still inside the control area after a step on the reading occurs. So low values of N, lead to high values of β and long delay times.

On the other hand, too high values of N produce a slow transition of the sample mean to its new value, what also increases the delay time. Thus, there must be an optimal value of N that minimizes the delay time of the test response.

Dependency of the delay time on σ_{proc}

In expresion (5.1) a confidence interval for the standard deviation of the process was established. Now, a specific value must be chosen as well to be used as estimation of σ. As the definition of $\hat{K}$ shows, the higher the estimation of σ is, the wider the control area results, and vice versa. So, if the value of the real standard deviation is higher than the estimated one, the number of *type I errors* will increase, thereby the estimated value must be high enough to respect the significance level α. On the other hand, if the estimated value is much higher than the real value, the control area will be too wide, leading to very long delay times.

[1]The delay time depending on σ, $\tilde{\sigma}$, μ and $\tilde{\mu}$ is calculated in the function "expectedtimeII.m" using "pascalemean.m" as well. Both are MATLAB functions. See appendix C.1

Thus, choosing the average value of the confidence interval of σ as estimation of the real value will lead to a test with a significance level $\alpha \leq 0.01$ just 50% of the time *(medium test)*. To ensure this significance level for a 97.5% of the cases, the highest value of the confidence interval in (5.1) should be used rather as estimation for σ *(wide test)*.

Figure 6.2 shows the average delay time for different values of N when the value of $\tilde{\sigma}_{proc}$ used to calculate K is set to the average value of the confidence interval (i.e.: $\tilde{\sigma}_{proc} = 194.3683$). In the same way, figure 6.3 shows the average delay time using the highest value of the confidence interval: $\tilde{\sigma}_{proc} = 266.7210$. Curves in both figures have a minimum as predicted.

As figures 6.2 and 6.3 show, the expected delay time is higher when using the wide test, but, any way, it is necessary to use it to ensure that the number of *tipe I errors* remains under control. Furthermore, even when we use the highest value of σ as estimation to calculate the test limits, it is possible that the real σ takes the lower values. Thus, taking $N = 10$ (minimizer of the red curve on figure 6.3) could lead to extremely long delay times if the real standard deviation takes low values (green curve). As a consequence, a value $N \geq 23$ must be used despite the average delay time becomes higher for lower values of σ (red and blue lines).

In conclusion, the highest value of σ must be used to calculate $K = \hat{K}\tilde{\sigma}$ and the lowest must be considered to choose N.

Finally, the test parameters in our case are:

$$\boxed{\begin{gathered} \tilde{\sigma}_{proc} = 266.7210, \quad N = 23 \\ \tilde{\sigma}_y = \frac{\tilde{\sigma}_{proc}}{\sqrt{N}} = 55.6151 \\ CL = \tilde{\mu}_y \pm \hat{K}\tilde{\sigma}_y = \tilde{\mu}_y \pm 185.8547 \end{gathered}} \tag{6.9}$$

It must be noted that the delay time is actually a statistical variable and should be described by a PDF, in spite of the fact that here it is treated within a deterministic framework. This means that delay time values higher than those predicted on figure 6.3 are actually expected to be obtained, while the predicted delay is just the average. This deterministic approach is justified by the fact that there is no specification for the maximal delay time for any significance level and by the fact that, as long as we minimize the average value, the maximal delay time is also minimized.

6.4 Results and discussion

The test has been implemented in function `"simulationSampleMean.m"` for MATLAB[2]. The function has been tried for its validation with real data sets including

[2] See appendix C.2

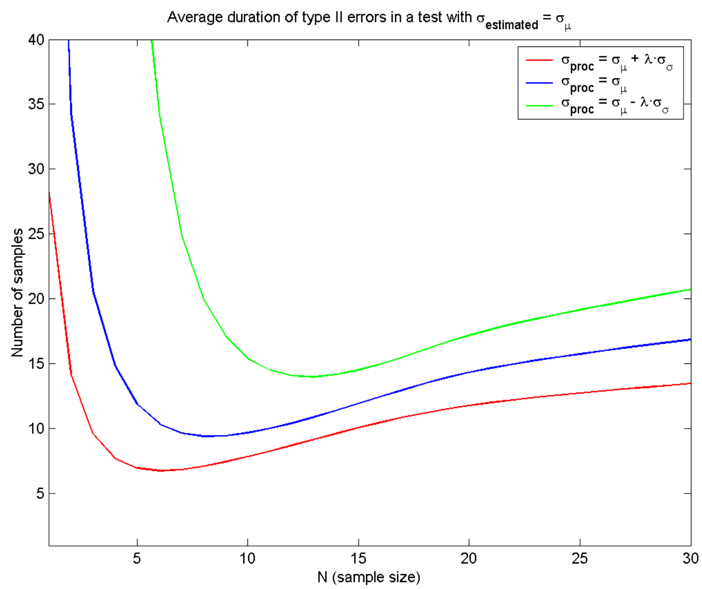


Figure 6.2: Delay Time vs. N with a medium test

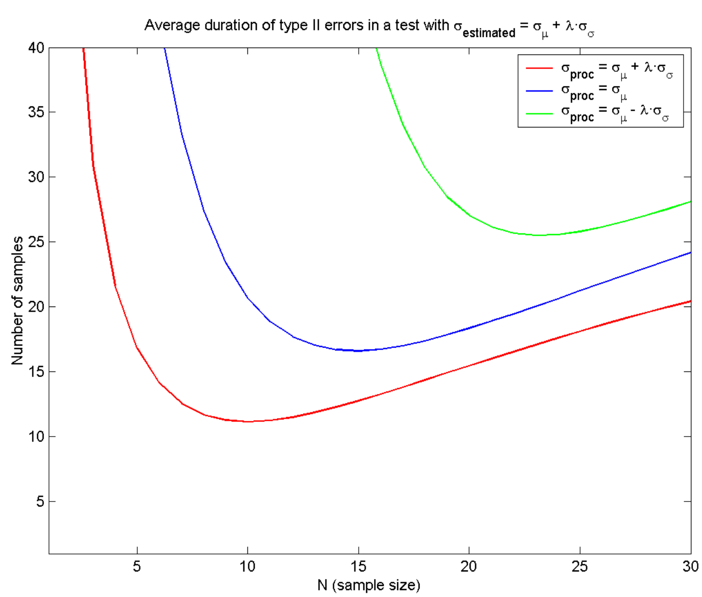


Figure 6.3: Delay Time vs. N with a wide test

some that were not used in the statistical analysis. Results can be seen on next figures including a label with information on the average real standard deviation of the process for each experiment. Figure 6.4 shows the response of the test when the real standard deviation of the process is relatively close to the estimated value $\tilde{\sigma}_{proc}$ used in (6.9) ($\sigma_{real} \approx 219$). Figure 6.5 shows the behavior of the test when the real standard deviation is much lower than the mentioned $\tilde{\sigma}_{proc}$.

Regardless of the good behavior of the test when the standard deviation of the process is close to the estimation used in the algorithm, results are not adequate for the rest of situations. Delay times, even when limited, are too long in many occasions for low real standard deviation values. In conclusion, the test would be suitable for this application if the range of possible standard deviation values were not so wide. Next chapter introduces a modification to solve this problem trough an on-line estimation of the standard deviation and adapting the test parameters to this temporary value.

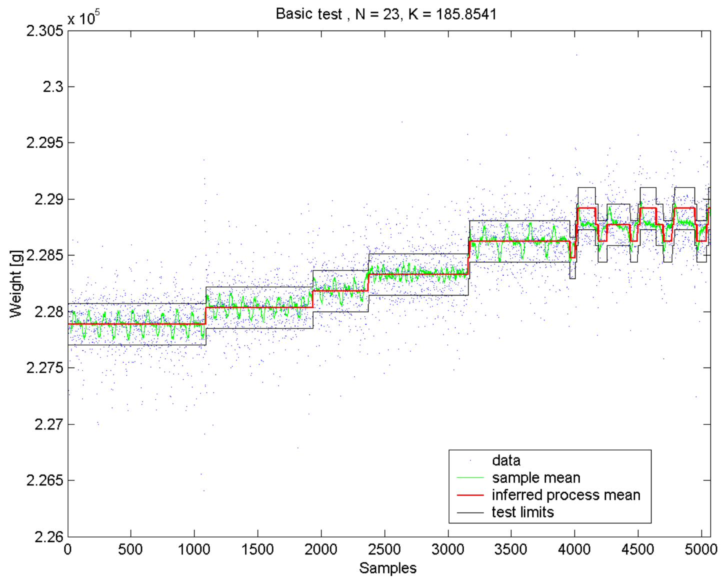


Figure 6.4: Basic algorithm applied to a data set with high standard deviation.

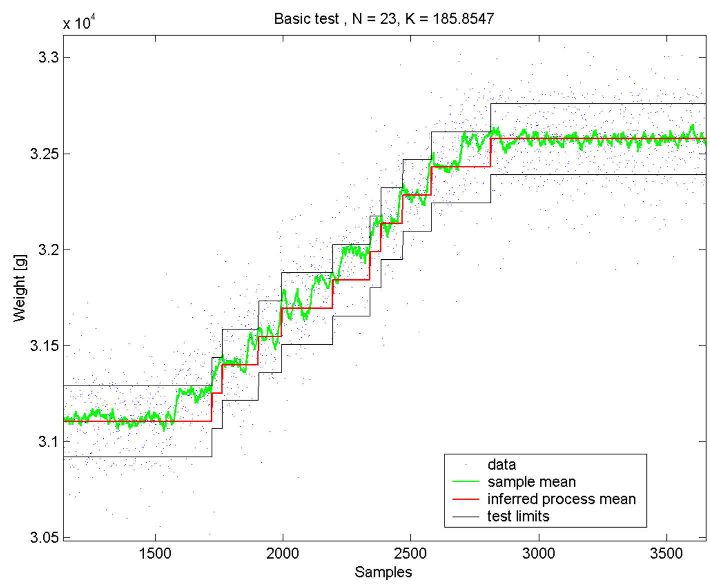


Figure 6.5: Basic algorithm applied to a data set with low standard deviation results in long delay times