Appendix A

Evolution of the expected value of $S^2(t)$ after a leap of the mean value

Let us consider a stochastic process whose parameters μ and σ change suddenly in a concrete moment. May $Y = X(\mu_1, \sigma_1)$ and $Z = X(\mu_2, \sigma_2)$ be the distributions of the process before and after the leap respectively. The variable x_i can be defined by

$$x_i = \begin{cases} y_i, & \text{if } i \leqslant a; \\ z_i, & \text{if } i > a. \end{cases} \tag{A.1}$$

where a is the last sample before the leap.

Let us consider a sample $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ of size N. The expression of his mean value $\bar{x}$ can be written as follows:

$$\bar{x} = \frac{a}{N}\bar{y} + \frac{(N-a)}{N}\bar{z} \tag{A.2}$$

Parameter a indicates the number of elements x_i in $\boldsymbol{x}$ corresponding to the the first distribution Y. As a takes values from N to 0, the expected value of $\bar{x}$ evolves linearly from μ_1 to μ_2.

Since in this case we are *not* dealing with a stationary process, the expected value of the *sample variance* is no longer the variance of the population ($E(S^2) \neq \sigma^2$). The new expected value is to be calculated here depending on a as a parameter. As known, the expression of the sample variance is

$$S_a^2 \doteq \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 \tag{A.3}$$

Considering (A.1) we can write:

$$S_a^2 = \frac{1}{N-1}\left(\sum_{i=1}^{a}(y_i-\bar{x})^2 + \sum_{i=a+1}^{N}(z_i-\bar{x})^2\right) \tag{A.4}$$

Adding and deducting μ in both addends and expanding the squares:

$$\begin{aligned}\sum_{i=1}^{a}(y_i-\bar{x})^2 &= \sum_{i=1}^{a}\left((y_i-\mu_1)-(\bar{x}-\mu_1)\right)^2 = \\ &= \sum_{i=1}^{a}\left((y_i-\mu_1)^2+(\bar{x}-\mu_1)^2\right) - 2\sum_{i=1}^{a}\left((y_i-\mu_1)(\bar{x}-\mu_1)\right) \qquad (A.5)\\ \sum_{i=a+1}^{N}(z_i-\bar{x})^2 &= \sum_{i=a+1}^{N}\left((z_i-\mu_2)-(\bar{x}-\mu_2)\right)^2 = \\ &= \sum_{i=a+1}^{N}\left((z_i-\mu_2)^2+(\bar{x}-\mu_2)^2\right) - 2\sum_{i=a+1}^{N}\left((z_i-\mu_2)(\bar{x}-\mu_2)\right) \qquad (A.6)\end{aligned}$$

In order to find the expected value of (A.4), the expected value of the terms in (A.5) and (A.6) are calculated.
For equation (A.5):

$$E\left[(y_i-\mu_1)^2\right] = \sigma_1^2 \tag{A.7}$$

Equation (A.2) is used now for the expansion of the next two terms

$$\begin{aligned}&E\left[(\bar{x}-\mu_1)^2\right] = \\ &= E\left[\left(\frac{a}{N}\cdot\bar{y}+\frac{(N-a)}{N}\cdot\bar{z}-\mu_1\right)^2\right] = E\left[\left(\frac{a}{N}(\bar{y}-\mu_1)+\frac{(N-a)}{N}(\bar{z}-\mu_1)\right)^2\right] = \\ &= \frac{a^2}{N^2}E\left[(\bar{y}-\mu_1)^2\right] + \frac{(N-a)^2}{N^2}E\left[(\bar{z}-\mu_1)^2\right] + 2\frac{a(N-a)}{N^2}E\left[(\bar{y}-\mu_1)(\bar{z}-\mu_1)\right] = \\ &= \frac{a^2}{N^2}\frac{\sigma_1^2}{a} + \frac{(N-a)^2}{N^2}E\left[\left((\bar{z}-\mu_2)+(\mu_2-\mu_1)\right)^2\right] + 0 = \\ &= \frac{a^2}{N^2}\frac{\sigma_1^2}{a} + \frac{(N-a)^2}{N^2}E\left[(\bar{z}-\mu_2)^2+(\mu_2-\mu_1)^2+2(\bar{z}-\mu_2)(\mu_2-\mu_1)\right] = \\ &= \frac{a}{N^2}\sigma_1^2 + \frac{(N-a)^2}{N^2}\left(\frac{\sigma_2^2}{(N-a)}+\Delta\mu^2+0\right) = \\ &= \frac{a}{N^2}\sigma_1^2 + \frac{(N-a)}{N^2}\sigma_2^2 + \frac{(N-a)^2}{N^2}\Delta\mu^2 \qquad (A.8)\end{aligned}$$

$$E\left[\sum_{i=1}^{a}((y_i-\mu_1)(\bar{x}-\mu_1))\right]=$$
$$=a\cdot E\left[(\bar{y}-\mu_1)(\bar{x}-\mu_1)\right]=a\cdot E\left[(\bar{y}-\mu_1)\left(\frac{a}{N}(\bar{y}-\mu_1)+\frac{(N-a)}{N}(\bar{z}-\mu_1)\right)\right]=$$
$$=\frac{a}{N}E\left[a(\bar{y}-\mu_1)^2+(N-a)(\bar{y}-\mu_1)(\bar{z}-\mu_1)\right]=\frac{a}{N}\left(\sigma_1^2+0\right)=$$
$$=\frac{a}{N}\sigma_1^2 \tag{A.9}$$

Doing the same with terms on equation (A.6):

$$E\left[(z_i-\mu_2)^2\right]=\sigma_2^2 \tag{A.10}$$

$$E\left[(\bar{x}-\mu_2)^2\right]=\frac{a}{N^2}\sigma_1^2+\frac{(N-a)}{N^2}\sigma_2^2+\frac{a}{N^2}\Delta\mu^2 \tag{A.11}$$

$$E\left[\sum_{i=a+1}^{N}((z_i-\mu_2)(\bar{x}-\mu_2))\right]=\frac{(N-a)}{N}\sigma_2^2 \tag{A.12}$$

Considering equations (A.7), (A.8), (A.9), (A.10), (A.11) and (A.12) the expected value of S_a^2 results:

$$E\left[S_a^2\right]=\frac{a}{N}\sigma_1^2+\frac{(N-a)}{N}\sigma_2^2+\frac{a(N-a)}{N(N-1)}\Delta\mu^2 \tag{A.13}$$