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Non-cooperative Rendezvous and Interception —A Direct Parametric Control Approach*

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***Abstract*— In this paper, the dynamical model in a matrix second-order nonlinear form is firstly established in the Sight of Line coordinate system for the problems of non-cooperative rendezvous and interception of spacecrafts, which are complete in the sense that no approximation is taken. Then, with the help of a recently proposed general parametric design approach for general fully-actuated second-order nonlinear systems, a direct parametric approach for spacecraft noncooperative rendezvous and interception control via proportional plus derivative feedback is proposed, which gives a complete parametrization of the pair of feedback gains, and allows usage of the established complete model. The approach possesses two important features. Firstly, with the proposed controller parametrization, the spacecraft rendezvous and the interception systems, though highly nonlinear, can be turned into constant linear systems with desire eigenstructure. Secondly, in such a design there are still degrees of freedom which may be further utilized to improve the system performance. Examples are considered to demonstrate the use of the proposed approach.**

***Index Terms*— Noncooperative rendezvous, Spacecraft interception, Fully-actuated second-order systems, Direct parametric approach, Nonlinear systems.**

I. Introduction

Spacecraft rendezvous has remained a challenging problem for many years ([1]-[4]), and has attracted much attention (e.g., [5]-[17]). Reported results are mainly fall into two categories.

One category is cooperative rendezvous in which the orbital model of the target spacecraft is known ([5]-[10]). This type of work relies on the equation of the relative motion of the chaser and the target, which reduces to the well-known T-H equation when the chaser and the target are close to each other, or the well-known C-W equation when, in addition, the target is in a circular orbit.

In certain circumstances, spacecraft rendezvous or inception with a non-cooperative target spacecraft is required, and such a problem has also attracted much attention in recent years ([11]-[17]). The difficulty underlying in such a problem is that the orbital model of the target is not assumed to be known.

With most of the reported results, the approaches used are those for systems represented by first-order system models of the linear form (e.g., [5]-[8])

$$\left\{\begin{array}{l} \dot{x}=A\left(t\right)x+B\left(t\right)u \\ y=C\left(t\right)x+D\left(t\right)u \end{array}\right. , \tag{1}$$

or the nonlinear form (e.g., [9], [10])

$$\left\{\begin{array}{l} \dot{x}=f\left(x,u,t\right) \\ y=g\left(x,u,t\right) \end{array}\right. , \tag{2}$$

where x, y and u are the state, output and control vectors, respectively, while A, B, C and D are the system coefficient matrices, and $f\left(x,u,t\right)$ and $g\left(x,u,t\right)$ are some proper vector functions with respect to the state vector x, the input vector u, as well as the time t. The first-order linear and nonlinear representations (1) and (2) of dynamical systems have been considered to be universal for decades because almost all models which are not originally in these forms can be converted into one of these forms. Therefore, great efforts have been taken in developing control strategies for systems represented by these models, and regarding applications, the first thing to do is to derive a system model in the form of (1) or (2), without thinking of the advantage that the original high-order model may offer.

Generally speaking, for many practical systems, the dynamical models are originally in a second-order format, since many physical phenomena are really governed by the Newton's Law or the Kirchoff's Law. What is more, often it is easier and more convenient to find the controller for a second-order system (see, e.g., [18]-[21]). While when the system is converted into a first-order one, not only the physical meanings of the variables as well as the system coefficients vanish, but also the advantages in controller design no longer exist. Very recently, inspired by his earlier work on control of second and high-order linear systems (see, e.g., [18]-[21]) and solutions to second and high-order generalized Sylvester matrix equations (see, e.g., [22]-[23]), the author has proposed a direct parametric control approach

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for a type of general fully-actuated second-order nonlinear systems ([24]), which possesses several important features.

In this paper, the general spacecraft rendezvous problem is considered, in which the general nonlinear dynamical model is used. The system model is expressed in a matrix second-order nonlinear form. By applying the direct parametric control design approach proposed in ([24]), a direct parametric control approach for the general spacecraft rendezvous problem is proposed, which has the following several advantages:

- it results in a constant linear closed-loop system although the open-loop system is highly nonlinear;
- it ensures the existence of a stable matrix F which determines the desired closed-loop eigenstructure;
- it provides certain degrees of freedom represented by a matrix Z, which is shown to form a dense set in the parameter space;
- it also allows the matrix F to be taken as a partially free parameter which contributes another part of degrees of freedom.; and
- it gives complete degrees of freedom represented by Z and F, which can be well utilized to achieve additional system properties.

II. Spacecraft Rendezvous Model

In this section, we present the dynamical model of the rendezvous system of spacecrafts. It is assumed that the considered spacecrafts are all rigid bodies and are subject to only gravity and the active impulse, Furthermore, affect of self-turning and turning around the sun, as well as some other factors are not considered.

Let $\boldsymbol{R}_{ch}$ be the vector from the earth center to the chaser spacecraft and $|\boldsymbol{R}_{ch}|$ be the module of $\boldsymbol{R}_{ch}$; $\boldsymbol{a}^{ch}$ is the acceleration resulted by the propeller force, μ is the gravity constant. Then the equation for the chaser spacecraft is

$$\ddot{\boldsymbol{R}}_{ch} = -\mu \frac{\boldsymbol{R}_{ch}}{|\boldsymbol{R}_{ch}|^3} + \boldsymbol{a}^{ch} \quad (3)$$

Similarly, defining $\boldsymbol{R}_{ta}$ as the vector from the earth center to the target spacecraft, and $|\boldsymbol{R}_{ta}|$ as the module of $\boldsymbol{R}_{ta}$, then the equation for the target spacecraft is

$$\ddot{\boldsymbol{R}}_{ta} = -\mu \frac{\boldsymbol{R}_{ta}}{|\boldsymbol{R}_{ta}|^3} + \boldsymbol{a}^{ta} \quad (4)$$

Further, let $\boldsymbol{r}$ be the vector from the chaser to the target, then

$$\boldsymbol{r} = \boldsymbol{R}_{ta} - \boldsymbol{R}_{ch}$$

Thus it follows from (3) and (4) that, within the inertial coordinate system, the relative motion of the two spacecrafts are governed by

$$\ddot{\boldsymbol{r}} = -\mu \left(\frac{\boldsymbol{R}_{ta}}{|\boldsymbol{R}_{ta}|^3} - \frac{\boldsymbol{R}_{ch}}{|\boldsymbol{R}_{ch}|^3} \right) + \boldsymbol{a}^{ta} - \boldsymbol{a}^{ch} \quad (5)$$

A. *The Dynamical Model*

The problem under consideration involves two types of coordinate systems, one is the Earth Center Inertial Coordinate system $O_I X_I Y_I Z_I$, the other is the Line of Sight Coordinate system $O_l x_l y_l z_l$.

The Earth Center Inertial Coordinate system (ECI) defines the center of mass of the earth as its origin; the Z_I axis is the axis of rotation of the earth in a positive direction, which intersects the celestial sphere at the celestial pole; and the X_I-Y_I plane of this coordinate system is taken as the equatorial plane of the earth, which is perpendicular to the earth's axis of rotation and the X_I is directed from the origin along the vernal equinox.

The Line of Sight Coordinate system (LOS) defines the center of mass of the chaser spacecraft as its origin; the x_l axis is directed from the origin to the target spacecraft; the axis y_l is on the x_l-Y_I plane and normal to the axis x_l; the z_l axis completes a right handed orthogonal frame (see Figure 1).

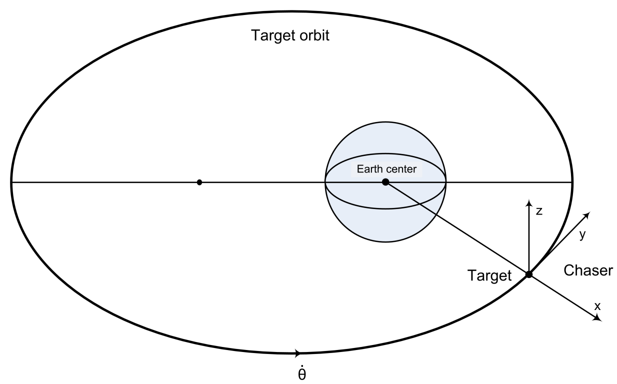

Fig. 1. Coordinate Systems

The transformation matrix from ECI to LOS can be shown to be as follows:

$$\begin{aligned} T_{lI} &= T_z(\varepsilon) T_y(\beta) \\ &= \begin{bmatrix} \cos\varepsilon & \sin\varepsilon & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos\varepsilon\cos\beta & \sin\varepsilon & -\cos\varepsilon\sin\beta \\ -\sin\varepsilon\cos\beta & \cos\varepsilon & \sin\varepsilon\sin\beta \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \end{aligned} \quad (6)$$

where the declination angle ε is measured from the axis x_l to the projection of x_l onto the X_I-Z_I plane; the drift angle β is measured from the axis X_I to the the projection of x_l onto the X_I-Z_I plane. Due to practical senses, the following assumption is required:

Assumption A0 $\varepsilon \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\beta \in (-\pi, \pi)$.

As ρ_r denotes the projection of r in the LOS frame, then, using the definition of cross production of vectors, we have

$$\ddot{r} = \ddot{\rho}_r + 2\omega \times \dot{\rho}_r + \omega \times (\omega \times \rho_r) + \dot{\omega} \times \rho_r \tag{7}$$

where ω and $\dot{\omega}$ are the angular velocity and the angular acceleration, respectively.

In the Rendezvous and interception problems, it is generally acknowledged that $|R_{ch}| \gg |r|$, so

$$\begin{aligned}
&\frac{R_{ch}+r}{|R_{ch}+r|^3} \\
=\;& \frac{R_{ch}+r}{|R_{ch}|^3}\left(1+2\frac{R_{ch}^{\mathrm{T}}r}{|R_{ch}|^2}+\frac{|r|^2}{|R_{ch}|^2}\right)^{-\frac{3}{2}} \\
\approx\;& \frac{R_{ch}+r}{|R_{ch}|^3}\left[1-\frac{3}{2}\left(2\frac{R_{ch}^{\mathrm{T}}r}{|R_{ch}|^2}+\frac{|r|^2}{|R_{ch}|^2}\right)\right] \\
\approx\;& \frac{1}{|R_{ch}|^3}\left(R_{ch}+r-3\frac{R_{ch}^{\mathrm{T}}r}{|R_{ch}|^2}R_{ch}\right),
\end{aligned}$$

therefore,

$$\frac{R_{ch}+r}{|R_{ch}+r|^3}-\frac{R_{ch}}{|R_{ch}|^3} \approx \frac{1}{|R_{ch}|^3}\left(r-3\frac{R_{ch}^{\mathrm{T}}r}{|R_{ch}|^2}R_{ch}\right).$$

Substituting the above relation into (5), yields

$$\begin{aligned}
\ddot{r} &= -\frac{\mu}{|\boldsymbol{R}_{ch}|^3}\left(r-3\frac{R_{ch}^{\mathrm{T}}r}{|R_{ch}|^2}R_{ch}\right)+a^{ta}-a^{ch} \\
&= -\xi(t)+a^{ta}-a^{ch}
\end{aligned} \tag{8}$$

where

$$\xi(t) = \frac{\mu}{|\boldsymbol{R}_{ch}|^3}\left(r-3\frac{R_{ch}^{\mathrm{T}}r}{|R_{ch}|^2}R_{ch}\right).$$

Denote

$$R_{ch} = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \end{bmatrix},$$

then we have

$$\begin{cases}
\xi_x = \frac{\mu}{|\boldsymbol{R}_{ch}|^3}\left[\left(\frac{3R_x^2}{R_x^2+R_y^2+R_z^2}-1\right)\rho\right] \\
\xi_y = \frac{\mu}{|\boldsymbol{R}_{ch}|^3}\left(\frac{3R_xR_y}{R_x^2+R_y^2+R_z^2}\rho\right) \\
\xi_z = \frac{\mu}{|\boldsymbol{R}_{ch}|^3}\left(\frac{3R_xR_z}{R_x^2+R_y^2+R_z^2}\rho\right).
\end{cases} \tag{9}$$

According to the above set of formulas, $\xi(t)$ can be treated as a measurable variable.

Combining (7) and (8), gives, within the line of sight coordinate system, the relative motion of the two spacecrafts as

$$\begin{aligned}
&\ddot{\rho}_r + 2\omega \times \dot{\rho}_r + \omega \times (\omega \times \rho_r) + \dot{\omega} \times \rho_r \\
&\quad + \xi(t) = a^{ta} - a^{ch}.
\end{aligned} \tag{10}$$

In the LOS frame, we have

$$\rho_r = \begin{bmatrix} \rho \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\rho}_r = \begin{bmatrix} \dot{\rho} \\ 0 \\ 0 \end{bmatrix}, \quad \ddot{\rho}_r = \begin{bmatrix} \ddot{\rho} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\omega &= T_z(\varepsilon)\begin{bmatrix} 0 \\ \dot{\beta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\varepsilon} \end{bmatrix} \\
&= \begin{bmatrix} \cos\varepsilon & \sin\varepsilon & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\beta} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \dot{\beta}\sin\varepsilon \\ \dot{\beta}\cos\varepsilon \\ \dot{\varepsilon} \end{bmatrix},
\end{aligned}$$

and

$$\dot{\omega} = \begin{bmatrix} \ddot{\beta}\sin\varepsilon + \dot{\beta}\dot{\varepsilon}\cos\varepsilon \\ \ddot{\beta}\cos\varepsilon - \dot{\beta}\dot{\varepsilon}\sin\varepsilon \\ \ddot{\varepsilon} \end{bmatrix},$$

then the left hand side of the equation (10) can be written as

$$\begin{aligned}
&\begin{bmatrix} \ddot{\rho} \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 & * & * \\ \dot{\varepsilon} & 0 & * \\ -\dot{\beta}\cos\varepsilon & * & 0 \end{bmatrix}\begin{bmatrix} \dot{\rho} \\ 0 \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & -\dot{\varepsilon} & \dot{\beta}\cos\varepsilon \\ \dot{\varepsilon} & 0 & -\dot{\beta}\sin\varepsilon \\ -\dot{\beta}\cos\varepsilon & \dot{\beta}\sin\varepsilon & 0 \end{bmatrix} \\
&\times \begin{bmatrix} 0 & * & * \\ \dot{\varepsilon} & 0 & * \\ -\dot{\beta}\cos\varepsilon & * & 0 \end{bmatrix}\begin{bmatrix} \rho \\ 0 \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & * & * \\ \ddot{\varepsilon} & 0 & * \\ -\ddot{\beta}\cos\varepsilon + \dot{\beta}\dot{\varepsilon}\sin\varepsilon & * & 0 \end{bmatrix}\begin{bmatrix} \rho \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \ddot{\rho} - \rho\left(\dot{\varepsilon}^2 + \dot{\beta}^2\cos^2\varepsilon\right) \\ \rho\ddot{\varepsilon} + 2\dot{\rho}\dot{\varepsilon} + \rho\dot{\beta}^2\sin\varepsilon\cos\varepsilon \\ -\rho\ddot{\beta}\cos\varepsilon - 2\dot{\rho}\dot{\beta}\cos\varepsilon + 2\rho\dot{\varepsilon}\dot{\beta}\sin\varepsilon \end{bmatrix}.
\end{aligned}$$

Thus the equations of relative motion (10) in the LOS frame can be expressed as follows:

$$\begin{cases}
\ddot{\rho} - \rho\left(\dot{\varepsilon}^2 + \dot{\beta}^2\cos^2\varepsilon\right) + \xi_x(t) = u_x \\
\rho\ddot{\varepsilon} + 2\dot{\rho}\dot{\varepsilon} + \rho\dot{\beta}^2\sin\varepsilon\cos\varepsilon + \xi_y(t) = u_y \\
-\rho\ddot{\beta}\cos\varepsilon - 2\dot{\rho}\dot{\beta}\cos\varepsilon + 2\rho\dot{\varepsilon}\dot{\beta}\sin\varepsilon + \xi_z(t) = u_z
\end{cases} \tag{11}$$

where

$$u = a^{ta} - a^{ch} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}.$$

When the angles ε and β are small enough, the above equation (11) becomes the following:

$$\begin{cases} \ddot{\rho} - \rho\left(\dot{\varepsilon}^2 + \dot{\beta}^2\right) + \xi_x(t) = u_x \\ \rho\ddot{\varepsilon} + 2\dot{\rho}\dot{\varepsilon} + \xi_y(t) = u_y \\ -\rho\ddot{\beta} - 2\dot{\rho}\dot{\beta} + \xi_z(t) = u_z. \end{cases}$$

B. The Matrix Second-order Form

Denote

$$q = \begin{bmatrix} \rho \\ \varepsilon \\ \beta \end{bmatrix}.$$

Then the dynamical model represented by (11) can be rewritten in the following matrix second-order form:

$$M(q)\ddot{q} + D(q,\dot{q})\dot{q} + K(q,\dot{q})q + \xi(t) = u, \tag{12}$$

where

$$M(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & -\rho\cos\varepsilon \end{bmatrix}, \tag{13}$$

$$D(q,\dot{q}) = \begin{bmatrix} 0 & 0 & 0 \\ 2\dot{\varepsilon} & 0 & 0 \\ -2\dot{\beta}\cos\varepsilon & 0 & 0 \end{bmatrix}, \tag{14}$$

and

$$K(q,\dot{q}) = \begin{bmatrix} -\dot{\varepsilon}^2 - \dot{\beta}^2\cos^2\varepsilon & 0 & 0 \\ \dot{\beta}^2\sin\varepsilon\cos\varepsilon & 0 & 0 \\ 2\dot{\varepsilon}\dot{\beta}\sin\varepsilon & 0 & 0 \end{bmatrix}. \tag{15}$$

C. The Interception Model

In the problem of interception, we do not care about the variable ρ, but only the direction determined by the angles β and ε, that is,

$$x = \begin{bmatrix} \varepsilon \\ \beta \end{bmatrix}.$$

With this consideration, let us drop the first equation in the rendezvous dynamical model (11), and obtain the following dynamical model for spacecraft interception:

$$\begin{cases} \rho\ddot{\varepsilon} + 2\dot{\rho}\dot{\varepsilon} + \rho\dot{\beta}^2\sin\varepsilon\cos\varepsilon + \xi_y(t) = u_y \\ -\rho\ddot{\beta}\cos\varepsilon - 2\dot{\rho}\dot{\beta}\cos\varepsilon + 2\rho\dot{\varepsilon}\dot{\beta}\sin\varepsilon + \xi_z(t) = u_z. \end{cases} \tag{16}$$

If we denote

$$u_I = \begin{bmatrix} u_y \\ u_z \end{bmatrix},\ \xi_I = \begin{bmatrix} \xi_y \\ \xi_z \end{bmatrix},$$

then the second-order form of the above spacecraft interception model (16) can be given as follows:

$$M_I(x)\ddot{x} + D_I(x,\dot{x})\dot{x} + \xi_I(t) = u_I, \tag{17}$$

with

$$M_I(x) = \begin{bmatrix} \rho & 0 \\ 0 & -\rho\cos\varepsilon \end{bmatrix}, \tag{18}$$

and

$$D_I(x,\dot{x}) = \begin{bmatrix} 2\dot{\rho} & \rho\dot{\beta}\sin\varepsilon\cos\varepsilon \\ 2\rho\dot{\beta}\sin\varepsilon & -2\dot{\rho}\cos\varepsilon \end{bmatrix}. \tag{19}$$

Please note that in this model, ρ and $\dot{\rho}$ are taken to be measured variables.

III. General Direct Parametric Approach

In this section, let us introduce the general direct parametric approach for fully-actuated second-order nonlinear systems, which is proposed in [24].

A. The Problem

Consider a type of systems in the following form:

$$A_2(\theta,x,\dot{x})\ddot{x} + A_1(\theta,x,\dot{x})\dot{x} + A_0(\theta,x,\dot{x})x + \xi(\theta,x,\dot{x}) = B(\theta,x,\dot{x})u \tag{20}$$

where $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}^n$ is the control vector, $\theta = \theta(t) \in \mathbb{R}^l$ is a parameter vector which satisfies the following assumption.

Assumption A1 The values of the system parameter $\theta = \theta(t) \in \mathbb{R}^l$ are within some compact set Ω, that is,

$$\theta(t) \in \Omega \subset \mathbb{R}^l.$$

The matrices $A_2(\theta,x,\dot{x})$, $A_1(\theta,x,\dot{x})$, $A_0(\theta,x,\dot{x}) \in \mathbb{R}^{n\times n}$ and $B(\theta,x,\dot{x}) \in \mathbb{R}^{n\times r}$ are the system coefficient matrices which are piece-wise continuous matrix functions with respect to $x, \dot{x}$ and θ. Note that in many applications, the coefficient matrix $A_2(\theta,x,\dot{x})$ is symmetric positive definite, here for convenience we impose the following so-called normality assumption.

Assumption A2 $\det A_2(\theta,x,\dot{x}) \neq 0, \quad \forall x, \dot{x}$ and $\theta(t) \in \Omega$.

The vector $\xi(\theta,x,\dot{x}) \in \mathbb{R}^n$ is a piece-wise continuous matrix function with respect to $x, \dot{x}$ and θ.

Furthermore, as the full-actuation requirement, we also require the following

Assumption A3 $r = n$ and $\det B(\theta,x,\dot{x}) \neq 0, \quad \forall x, \dot{x}$ and $\theta(t) \in \Omega$.

For control of the above system (20), we intend to design a controller which is composed of two parts:

$$u = u_c + u_f, \tag{21}$$

where u_c compensates the term $\xi(\theta,x,\dot{x})$ in the system model, and is simply given by

$$u_c = B^{-1}(\theta,x,\dot{x})\xi(\theta,x,\dot{x}); \tag{22}$$

while u_f is a proportional plus derivative state feedback in the following form:

$$\begin{aligned} u_f &= K_0(\theta,x,\dot{x})x + K_1(\theta,x,\dot{x})\dot{x} + v \\ &= [K_0(\theta,x,\dot{x}) \quad K_1(\theta,x,\dot{x})]\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + v, \end{aligned} \tag{23}$$

where $K_0(\theta, x, \dot{x}), K_1(\theta, x, \dot{x}) \in \mathbb{R}^{n\times n}$ are the feedback gains to be designed, which are piece-wisely continuous functions with respect to $x, \dot{x}$ and θ, and v is an external signal. With this controller applied to the fully-actuated system (20), the closed-loop system is obviously obtained as follows:

$$A_2(\theta, x, \dot{x})\ddot{x} + A_1^c(\theta, x, \dot{x})\dot{x} + A_0^c(\theta, x, \dot{x})x = B(\theta, x, \dot{x})v, \tag{24}$$

where

$$\begin{cases} A_0^c(\theta, x, \dot{x}) = A_0(\theta, x, \dot{x}) - B(\theta, x, \dot{x})K_0(\theta, x, \dot{x}) \\ A_1^c(\theta, x, \dot{x}) = A_1(\theta, x, \dot{x}) - B(\theta, x, \dot{x})K_1(\theta, x, \dot{x}) \end{cases}. \tag{25}$$

If we let

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

then, in view of Assumption A3, the closed-loop system (24)-(25) can also be converted into the following first-order form:

$$\dot{X} = A_c(\theta, x, \dot{x})X + B_c(\theta, x, \dot{x})v, \tag{26}$$

with

$$A_c(\theta, x, \dot{x}) = \begin{bmatrix} 0 & I_n \\ -A_2^{-1}(\theta, x, \dot{x})A_0^c & -A_2^{-1}(\theta, x, \dot{x})A_1^c \end{bmatrix}, \tag{27}$$

$$B_c(\theta, x, \dot{x}) = \begin{bmatrix} 0 \\ B(\theta, x, \dot{x}) \end{bmatrix}. \tag{28}$$

and our design purpose is to let $A_c(\theta, x, \dot{x})$ be similar to an arbitrary given constant matrix of the same dimension as stated in the following problem.

Problem FA Given the system (20) satisfying Assumptions A1-A3, and an arbitrarily chosen matrix $F \in \mathbb{R}^{2n\times 2n}$, find a constant nonsingular matrix $V \in \mathbb{R}^{2n\times 2n}$, and a pair of gain matrices $K_0(\theta, x, \dot{x})$ and $K_1(\theta, x, \dot{x}) \in \mathbb{R}^{n\times n}$, such that,

$$V^{-1}A_c(\theta, x, \dot{x})V = F. \tag{29}$$

As a consequence, the closed-loop system matrix

$$A_c(\theta, x, \dot{x}) = VFV^{-1} \tag{30}$$

is a constant one. Such a requirement is generally difficult to achieve for a nonlinear system, but we will show in the sequential of the paper that this is actually achievable under the full-actuation assumption A3.

B. The Direct Parametric Approach

Define

$$\mathbb{F} = \left\{F \mid F \in \mathbb{R}^{2n\times 2n}, \text{ and } \exists Z \in \mathbb{R}^{n\times 2n}, \text{ s.t. } \det\begin{bmatrix} Z \\ ZF \end{bmatrix} \neq 0\right\},$$

then the following result gives a complete answer to Problem FA ([24]).

Theorem 1: Problem FA has a solution if and only if $F \in \mathbb{F}$, and in this case all the solutions to Problem FA are parametrized as

$$V = V(Z, F) = \begin{bmatrix} Z \\ ZF \end{bmatrix}, \tag{31}$$

and

$$\begin{aligned} &[K_0(\theta, x, \dot{x}) \quad K_1(\theta, x, \dot{x})] \\ = \; & B^{-1}(\theta, x, \dot{x})W(\theta, x, \dot{x}, Z, F)V(Z, F)^{-1}, \end{aligned} \tag{32}$$

with

$$\begin{aligned} &W(\theta, x, \dot{x}, Z, F) \\ = \; & A_2(\theta, x, \dot{x})ZF^2 + A_1(\theta, x, \dot{x})ZF + A_0(\theta, x, \dot{x})Z, \end{aligned} \tag{33}$$

where $Z \in \mathbb{R}^{n\times 2n}$ is an arbitrary parameter matrix satisfying

$$\det\begin{bmatrix} Z \\ ZF \end{bmatrix} \neq 0. \tag{34}$$

The following further addresses some related issues.

It follows from Theorem 1 that the solvability of Problem FA depends on the nonemptyness of the set $\mathbb{F}$. Thus conditions for the nonemptyness of the set $\mathbb{F}$ arise to be a very closely related issue. The following result gives the condition for the set $\mathbb{F}$ to be nonempty ([24]).

Theorem 2: $\mathbb{F}$ is nonempty if and only if there exists a nonsingular matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$Q^{-1}FQ = J_F = \text{Blockdiag}(J_1, J_2), \tag{35}$$

with $J_1, J_2 \in \mathbb{R}^{n\times n}$ having no common eigenvalues, that is,

$$\sigma(J_1) \cap \sigma(J_2) = \phi. \tag{36}$$

Due to Theorem 2, let us define

$$\begin{aligned} \mathbb{F}_J = \; \{F \mid \; & F \in \mathbb{R}^{2n\times 2n}, \text{ and } \exists Q, J_1, J_2 \in \mathbb{R}^{n\times n}, \text{s.t} \\ & \det Q \neq 0, \; \sigma(J_1) \cap \sigma(J_2) = \phi \\ & \text{and } Q^{-1}FQ = \text{Blockdiag}(J_1, J_2)\}. \end{aligned}$$

It thus follows from Theorem 2 that $\mathbb{F}_0 = \mathbb{F} \cap \mathbb{F}_J$ is always nonempty.

The second related issue is about the parameter matrix Z. Define for matrix $F \in \mathbb{R}^{2n\times 2n}$ the associated set

$$\mathbb{Z}_0(F) = \left\{Z \mid Z \in \mathbb{R}^{n\times 2n}, \text{ and } \det\begin{bmatrix} Z \\ ZF \end{bmatrix} \neq 0\right\},$$

then, it is clearly observed that this is the set of the free parameter matrix Z in the proposed direct parametric design. As implied by Theorem 2, $\mathbb{Z}_0(F)$ is nonempty when $F \in \mathbb{F}_0$. While as a matter of fact this set is not only nonempty, but is also a Zariski open set in $\mathbb{R}^{n\times 2n}$, and hence is dense in $\mathbb{R}^{n\times 2n}$. The proof of such a conclusion is given in [24]. Such a result states that the condition (34) on the free parameter matrix Z is really not a strict one at all, it can be satisfied by almost all $Z \in \mathbb{R}^{n\times 2n}$ in the Zariski sense. Therefore, in many applications, this condition can be actually neglected in the optimization problem formed to optimize this free parameter Z. In practical applications, the parameter matrices F and Z should be optimized to improve the closed-loop system performance.

IV. Non-cooperative Rendezvous

In this section, we apply the general direct parametric approach for general nonlinear fully-actuated second-order systems to the spacecraft rendezvous model in the nonlinear matrix second-form (12)-(15). Regarding the three assumptions, A1-A3, required by the direct parametric approach on the system model, we have the following observations:

- Assumption A1 is met automatically since the parameter $\theta(t)$ does not exist in the spacecraft rendezvous system model (12)-(15);
- Assumption A2 holds since in the spacecraft rendezvous system model (12)-(15) we have under Assumption A0
$$\det M(q) = \rho^2 \cos\varepsilon \neq 0;$$
- Assumption A3 holds since in the spacecraft rendezvous system model (12)-(15) we have $B(q,\dot{q}) = I_3$.

It follows from the above last point that the system is indeed a fully-actuated one.

A. The Problem

Following the general direct parametric approach for control systems design, the controller to be designed for the spacecraft rendezvous system is composed of two parts:

$$u = u_c + u_f, \tag{37}$$

where u_c compensates the term $\xi(t)$ in the system model, and is simply given by

$$u_c = \xi(t); \tag{38}$$

while u_f is a proportional plus derivative state feedback in the following form:

$$\begin{aligned} u_f &= K_0(q,\dot{q})\,q + K_1(q,\dot{q})\,\dot{q} + v \\ &= \begin{bmatrix} K_0(q,\dot{q}) & K_1(q,\dot{q}) \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + v, \end{aligned} \tag{39}$$

where $K_0(q,\dot{q}), K_1(q,\dot{q}) \in \mathbb{R}^{3\times 3}$ are the feedback gains to be designed, and v is an external signal.

With this controller applied to the fully-actuated system (12), the closed-loop system is obviously obtained as

$$\ddot{q} + A_1^c(q,\dot{q})\,\dot{q} + A_0^c(q,\dot{q})\,q = v, \tag{40}$$

or

$$\dot{X} = A_c(q,\dot{q})\,X + B_c v, \tag{41}$$

with

$$X = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

$$A_c(q,\dot{q}) = \begin{bmatrix} 0 & I_n \\ -M^{-1}A_0^c(q,\dot{q}) & -M^{-1}A_1^c(q,\dot{q}) \end{bmatrix}, \tag{42}$$

$$B_c(q) = \begin{bmatrix} 0 \\ I_3 \end{bmatrix}, \tag{43}$$

and

$$\begin{cases} A_0^c(q,\dot{q}) = K(q,\dot{q}) - K_0(q,\dot{q}) \\ A_1^c(q,\dot{q}) = D(q,\dot{q}) - K_1(q,\dot{q}) \end{cases}. \tag{44}$$

Again, our design purpose is to let $A_c(q,\dot{q})$ to be similar to an arbitrary given constant matrix of the same dimension.

Problem NCR Given the rendezvous system model (12)-(15), and an arbitrarily chosen matrix $F \in \mathbb{R}^{6\times 6}$, find a constant nonsingular matrix $V \in \mathbb{R}^{6\times 6}$, and a pair of gain matrices $K_0(q,\dot{q})$ and $K_1(q,\dot{q}) \in \mathbb{R}^{3\times 3}$, such that,

$$V^{-1}A_c(q,\dot{q})\,V = F. \tag{45}$$

As a consequence of the requirement in the above problem, the closed-loop system matrix

$$A_c(q,\dot{q}) = VFV^{-1} \tag{46}$$

is a constant one.

B. Direct Parametric Approach

Applying the direct parametric approach to the spacecraft rendezvous system model (12)-(15), gives the following result.

Theorem 3: Problem NCR has a solution if and only if $F \in \mathbb{F}$, and in this case all the solutions to Problem FA are parametrized as

$$V = V(Z,F) = \begin{bmatrix} Z \\ ZF \end{bmatrix}, \tag{47}$$

and

$$\begin{aligned} &\begin{bmatrix} K_0(q,\dot{q}) & K_1(q,\dot{q}) \end{bmatrix} \\ =\ & G(q)\left[MZF^2 + DZF + KZ\right] V(Z,F)^{-1}, \end{aligned} \tag{48}$$

where $Z \in \mathbb{R}^{3\times 6}$ is an arbitrary parameter matrix satisfying

$$\det \begin{bmatrix} Z \\ ZF \end{bmatrix} \neq 0. \tag{49}$$

Based on Theorem 1, we can give a procedure for carrying out the direct parametric control design of the spacecraft rendezvous system model (12)-(15).

Step 1 *Defining the structure of matrix F*

The structure of the matrix F is usually in a Jordan form or a diagonal form. To make sure that it is Hurwitz, it is required that the eigenvalues of the matrix lie in the left hand complex plane, that is,

$$\lambda_i(F) \in \mathbb{C}^-,\ i = 1 \sim 6. \tag{50}$$

In certain cases, this matrix may be simply chosen to be a specific Hurwitz matrix.

Step 2 *Forming an optimization problem*

According to the system requirements, establish an index

$$J = J(F, Z),$$

which is a scalar function with respect to the design parameters F and Z, and then form an optimization problem of the following form:

$$\begin{array}{c} \min J(F, Z) \\ \text{s.t.} \quad (49), (50) \end{array}. \tag{51}$$

Depending on the specific problem, there may be other constraints added to the above optimization. Also, in many practical applications, the constraint (49) can be often neglected since it is satisfied for almost any matrix Z.

Step 3 *Seeking parameters*

Find the optimal (or sub-optimal) parameters F and Z by solving the above optimization problem (51) using some proper optimization algorithm.

Step 4 *Computing the controller gains*

Compute the controller gains according to the parametric expression of the feedback gains given in formulas (47)-(48). In certain cases the closed-loop eigenvector matrix V may also need to be obtained by the expression (31) and the closed-loop system matrix may be obtained as $A_c = VFV^{-1}$.

V. Spacecraft Interception

Very similarly, by applying the general direct parametric approach for general nonlinear fully-actuated second-order systems to the spacecraft interception model in the nonlinear matrix second-form (17)-(19), we can get a parametric solution to the spacecraft interception problem.

A. The Result

Again, the controller to be designed is composed of two parts:

$$u = u_c + u_f, \tag{52}$$

where u_c compensates the term $\xi_I(t)$ in the system model, and is simply given by

$$u_c = \xi_I(t); \tag{53}$$

while u_f is a proportional plus derivative state feedback in the following form:

$$\begin{aligned} u_f &= K_0(x, \dot{x})\, x + K_1(x, \dot{x})\, \dot{x} + v \\ &= [K_0(x, \dot{x}) \quad K_1(x, \dot{x})] \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + v, \end{aligned} \tag{54}$$

where $K_0(\theta, x, \dot{x}), K_1(\theta, x, \dot{x}) \in \mathbb{R}^{3\times 3}$ are the feedback gains which are given by

$$\begin{aligned} &[K_0(x, \dot{x}) \quad K_1(x, \dot{x})] \\ = &\left[M_I(x) ZF^2 + D_I(x, \dot{x}) ZF\right] V(Z, F)^{-1}, \end{aligned} \tag{55}$$

with

$$V = V(Z, F) = \begin{bmatrix} Z \\ ZF \end{bmatrix}, \tag{56}$$

where $F \in \mathbb{R}^{4\times 4}$ is a proper stable matrix, and $Z \in \mathbb{R}^{2\times 4}$ is an arbitrary parameter matrix satisfying

$$\det \begin{bmatrix} Z \\ ZF \end{bmatrix} \neq 0. \tag{57}$$

With this controller applied to the fully-actuated system (12), the closed-loop system is a constant linear one in the following form:

$$\dot{X} = \left(VFV^{-1}\right) X + \begin{bmatrix} 0 \\ I_2 \end{bmatrix} v, \tag{58}$$

where

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Based on the above result, a similar procedure for carrying out the direct parametric control of the spacecraft interception system (17)-(19) can also be given.

B. An Example

Let us design specifically a controller for the spacecraft interception problem following a procedure similar to that given in Section IV.

Step 1 Without loss of generality, let us take

$$F = \text{Blockdiag}\left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, -3, -4\right),$$

whose set of eigenvalues is

$$\sigma(F) = \{-1 \pm j, -3, -4\}.$$

Correspondingly, we have

$$J_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}.$$

Step 2 Due to space limitation, optimization of parameter Z is not considered.

Step 3 For simplicity, we just choose

$$Z = \begin{bmatrix} I_2 & I_2 \end{bmatrix}.$$

It can be easily checked that

$$V = \begin{bmatrix} I_2 & I_2 \\ J_1 & J_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & -3 & 0 \\ -1 & -1 & 0 & -4 \end{bmatrix}$$

is nonsingular, that is, the constraint (34) is met, and it can be computed that

$$V^{-1} = \frac{1}{7} \begin{bmatrix} 9 & -4 & 3 & -1 \\ 3 & 8 & 1 & 2 \\ -2 & 4 & -3 & 1 \\ -3 & -1 & -1 & -2 \end{bmatrix}.$$

Step 4 Note that

$$ZFV^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$ZF^2V^{-1} = \frac{1}{7}\begin{bmatrix} -24 & 20 & -29 & 5 \\ -30 & -24 & -10 & -34 \end{bmatrix},$$

we then have

$$M_I(x) = \begin{bmatrix} \rho & 0 \\ 0 & -\rho\cos\varepsilon \end{bmatrix}, \tag{59}$$

and

$$D_I(x,\dot{x})\,ZFV^{-1} = [0_{2\times 2} \quad D(x,\dot{x})].$$

It thus follows from (55)-(56) that the gain matrices are given by

$$K_0(x,\dot{x}) = \frac{\rho}{7}\begin{bmatrix} -24 & 20 \\ 30\cos\varepsilon & 24\cos\varepsilon \end{bmatrix},$$

and

$$\begin{aligned} & K_1(x,\dot{x}) \\ = \; & \frac{\rho}{7}\begin{bmatrix} -29 & 5 \\ 10\cos\varepsilon & 34\cos\varepsilon \end{bmatrix} + D(x,\dot{x}) \\ = \; & \frac{1}{7}\begin{bmatrix} 14\dot{\rho} - 29\rho & 7\rho\dot{\beta}\sin\varepsilon\cos\varepsilon + 5\rho \\ 14\rho\dot{\beta}\sin\varepsilon + 10\rho\cos\varepsilon & -14\dot{\rho}\cos\varepsilon + 34\rho\cos\varepsilon \end{bmatrix}. \end{aligned}$$

With the above designed controller, the closed-loop system can be checked to be

$$\dot{x} = A_c x + \begin{bmatrix} 0_{2\times 2} \\ I_2 \end{bmatrix} v,$$

with

$$A_c = \frac{1}{7}\begin{bmatrix} 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \\ -24 & 20 & -29 & 5 \\ -30 & -24 & -10 & -34 \end{bmatrix}.$$

VI. Conclusion

In this paper, direct parametric approaches for spacecraft noncooperative rendezvous control spacecraft interception control are proposed. Different from many previously reported result, the spacecraft rendezvous model and interception model are established in a second-order nonlinear format. It is shown that for these models simple controller parametrizations exist in the form of state proportional plus derivative feedback. An important consequence of the proposed set of controllers is that the resulted in closed-loop system is a linear constant one with designed eigenstructure which is determined by the chosen matrix F. Besides the matrix F, the truly parameter matrix existing in the parametric design is the matrix Z, which forms a dense set in the parameter space. In practical applications, it is suggested that the matrices F and Z are optimized simultaneously to achieve additional requirements on the closed-loop system.

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