# Chapter 3

# Multibody System Approach and Frequency Analysis Methods

This chapter shows the main ideas about the mathematical statement of a multibody system [7] and about the analysis in the frequency domain[3].

## 3.1 Kinematics and kinetics of multibody systems

The multibody system approach is related to the free body principle of one single body, Fig (3.1). The position of body $i$ is given by the 3x1-translation vector $r_{Ii}$ of an arbitrary body fixed point $O_i$ and by the 3x3-rotation tensor $S_{Ii}$ relating the body-fixed frame $i$ to the inertial frame $I$. Then, by differentiating, the absolute 3x1-translational and rotational acceleration vectors $a_{Ii}$ and $\alpha_{Ii}$, respectively, are obtained.

For the formulation of the dynamical Newton's and Euler's equations the body-fixed frame $i$ is more adequate. Therefore, the acceleration vectors will be resolved in frame $i$ and represented by the overall 6x1-acceleration vector

$$\ddot{x}_i = \begin{bmatrix} a_{Ii}^t S_{Ii} & \alpha_{Ii}^t S_{Ii} \end{bmatrix}^t \tag{3.1}$$

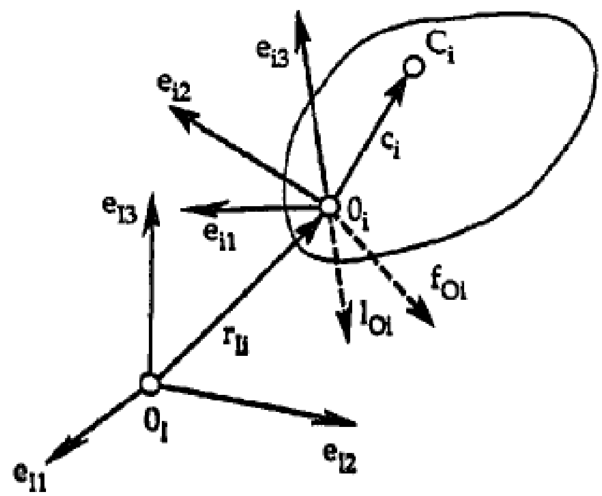


Figure 3.1: Free body of multibody system.

Then, Newton's and Euler's equations read as

$$M_i \ddot{x}_i + k_i = q_i, \quad i = 1, \ldots, p, \tag{3.2}$$

where $M_i$ is a time-invariant 6x6-inertia matrix composed of the masses $m_i$, the 3x1-vector $c_i$ between the reference point $O_i$ and the centre of mass $C_i$ and the 3x3-tensor of moments of inertia $I_{Oi}$. Further, the 6x1-vector $k_i$ represents the gyroscopic forces and $q_i = [\begin{array}{cc} f_{Oi}^t & l_{Oi}^t \end{array}]^t$ is the 6x1-vector summarizing the 3x1-force vector $f_{Oi}$ and the 3x1-torque vector $l_{Oi}$ also resolvec in the body-fixed frame $i$. Introducing $q$ constraints, the free system of $p$ bodies is assembled as a holonomic system. Then, all the kinematical quantities depend on the f = 6p-q generalized coordinates of the system represented by the fx1-position vector $y$. In particular, it yields for scleronomic systems

$$r_{Ii} = r_{Ii}(y), \quad S_{Ii} = S_{Ii}(y) \tag{3.3}$$

By differentiating with respect to the inertial frame considering the generalized coordinates and resolving in the body-fixed frame, one finally arrives at the 6x1-acceleration vector

$$\ddot{x}_i \overline{J}_i \ddot{y} + \overline{a}_i, \quad i = 1, \dots, p, \tag{3.4}$$

where the 6x1-Jacobian matrix $\overline{J}_i$ and a 3x1-vector $\overline{a}_i$ are introduced. Further, the reaction forces and torques $\overline{q}_i^r = [\ f_i^{rt} \quad l_i^{rt}\ ]^t$ have to be added:

$$q_i = \overline{q}_i^a + \overline{q}_i^r = \overline{q}_i^a + \overline{Q}_i g \tag{3.5}$$

Where $\overline{q}_i^a$ are the ramining applied forces, $\overline{Q}_i$ is a 6xq-distribution matrix and $g$ represents the qx1-vector of the generalized reactionforces. then, the Newton-Euler equations of the total system read as

$$\overline{\overline{M}}\ddot{x} + \overline{k} = \overline{q} + \overline{Q}g \tag{3.6}$$

where M = diag($M_i$) = const, $\overline{J}^t = [\ \overline{J}_1^t \quad \overline{J}_2^t \quad \dots \overline{J}_p^t\ ]$ and $\overline{Q}^t = [\ \overline{Q}_1^t \quad \overline{Q}_2^t \quad \dots \overline{Q}_p^t\ ]$ represent global matrices of the system.

For moderate sound pressure level, linear kinematics and linear constitutive equations of ligaments can be assumed. Then, the equation 3.6 can be linearized to perform eigenvalue analysis as a powerful tool for the investigations of the dynamical behavior.

$$x_s = x_{ssoll} + y_s \tag{3.7}$$

With the vector of small deformations defined around the static equilibrium position (see eq. 3.7) the equations of motion have the form

$$M\ddot{y}_s(t) + Q\dot{y}_s(t) + Ky_s(t) = h(t) \tag{3.8}$$

Where the mass matrix $M$ contains the mass and inertia properties of the multibody systems member, $Q$ contains the forces dependent on velocities, and forces dependent on displacements are described by the stiffness matrix $K$. The external excitation of the multibody system is summarized in vector $h$.

## 3.2 Analysis in the Frequency Domain.

For the simulation the linearized second order differential equation system (eq. 3.8 ) had to be transform in a first order ordinary differential equation system. Thus, a state vector is defined (eq. 3.9).

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \tag{3.9}$$

A first order linear differential equation system always can be expressed in a minimal form (eq. 3.10)

$$\dot{x}(t) = A(t)x(t) + b(t) \tag{3.10}$$

If the vector $b$ is null, the system is homogeneous and has the form of the equation 3.11.

$$\dot{x}(t) = A(t)x(t) \tag{3.11}$$

In the case of the homogeneous system, the vector $x$ has a linear relation with its derivative (eq. 3.12).

$$\dot{x} = \lambda x \tag{3.12}$$

Where $\lambda$ is one of the roots of the equation 3.13, with $E$ the identity matrix, and it is called eigenvalue of the matrix $A$.

$$det(\lambda E - A) = 0 \tag{3.13}$$

If $\lambda$ is an eigenvalue of the matrix $A$, then $\tilde{x}$ is called associate eigenvector to $\lambda$ to any no-null vector such that it is a solution of the equation 3.14

$$(\lambda E - A)\tilde{x} = 0 \tag{3.14}$$

Thus, one solution of the homogeneneous system can be expressed as a function of an eigenvalue $\lambda$ and its associate eigenvector $\tilde{x}$ as it is shown in the equation 3.15. This solution is called associate eigenfuction to $\lambda$ and $\tilde{x}$.

$$x(t) = e^{\lambda t}\tilde{x} \tag{3.15}$$

It is well-known that in order to solve a homogenneous first order linear differential equation system of $n$ equations and $n$ unknowns is necessary to find out $n$ linear-independent solutions of the system given by the equation 3.11 in the form:

$$x = \varphi(t) = \textstyle\sum_{i=1}^{n} c_i\varphi_i(t), \quad c_i \in C \quad 1 \leq i \leq n, \tag{3.16}$$

Since associate eigenvectors to different eigenvalues are linear-independent and associate eigenfunction to these eigenvectors are linear-independent as well, it is possible to express the solution of the system as a function of the its eigenvalues and eigenvectors (see eq. 3.17).

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\tilde{x}_i \tag{3.17}$$

If the system is no-homogeneous, that means, it has a vector *b(t)*, then it is necessary to find out a particular solution of the system in order to add it to the set of solutions of the homogeneous system. If the system has an excitation vector $b_h(t)$ such that it is a harmonic function of frequency $\Omega$ and each component $i$ has the form $b_{h_i}(t) = e_i cos(\Omega t - \phi_i)$; with $i = 1 \ldots n$. Where $e_i$ are the amplitudes of the signals and $\phi_i$ their phase lag. Thus, the system has the form:

$$\dot{x}(t) = A(t)x(t) + b_h(t) \tag{3.18}$$

For each value of the frequency $\Omega$ the vector $b_h(t)$ can be translate into the equivalent complex form $b_h^*(\Omega)$. The stationary response of the system is harmonic, as well, and it can be obtained in the frequency domain as the vector $g(\Omega$ (see eq. 3.19).

$$g(\Omega) = F(\Omega)b_h^*(\Omega) \tag{3.19}$$

Where the matrix $F(\Omega)$ is the frequency response matrix of dimensions $nxn$ and it can be calculated as it is shown in the equation 3.20.

$$F(\Omega) = (i\Omega E - A)^{-1} \tag{3.20}$$

As it was mentioned before, since the excitation signal is an harmonic function, the stationary output signal will be an harmonic function as well. Thus, the stationary response of the system will have the following form in the temporal domain:

$$x_{i\infty}(t) = a_i cos(\Omega t - \Psi_i) \tag{3.21}$$

Where that amplitude $a_i$ and the phase lag $\Psi_i$ are obtained of the frequency response vector $g(\Omega)$ as it is shown in the equations 3.22 and 3.23.

$$a_i = \sqrt{(g_{Re_i}(\Omega))^2 + (g_{Im_i}(\Omega))^2} \tag{3.22}$$

$$\Psi_i = arctan(\frac{g_{Im_i}(\Omega)}{g_{Re_i}(\Omega)}) \tag{3.23}$$

As it was indicated before, the frequency response vector $g(\Omega)$ depends on the matrix $A$, the frequency $\Omega$ and the excitation vector in its complex form $b_h^*(t)$. It is easy to come up to the expressions of the real and imaginary parts of the frequency response vector through the equations 3.19 and 3.20. The results are shown in the equations 3.24 and 3.25.

$$(\Omega E + A) g_{Re}(\Omega) = -A b_h^* \tag{3.24}$$

$$(\Omega E + A) g_{Im}(\Omega) = -\Omega b_h^* \tag{3.25}$$

Both equations can be expressed in an only system as it is done in the equation 3.26.

$$(\Omega E + A) \left[ \begin{array}{cc} g_{Re}(\Omega) & g_{Im}(\Omega) \end{array} \right] = \left[ \begin{array}{cc} -A b_h^* & -\Omega b_h^* \end{array} \right] \tag{3.26}$$